A New Axiomatic Approach to Index Number Theory

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APRIL 2004

Discussion Paper No.: 04-05

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Abstract

The paper develops a new axiomatic approach to bilateral index number theory that treats the prices and expenditure shares on N commodities for two periods as the independent variables in the index number formula instead of the traditional approach, which treats the two price vectors and the two quantity vectors as the independent variables. The approach was suggested over a century ago by Walsh (1901). A new axiomatic characterization of the Törnqvist (1936) Theil (1967) price index is obtained.

Key Words

Bilateral index number theory; the test approach to index number theory; the axiomatic approach to index number theory; the Törnqvist Theil price index.

Journal of Economic Literature Classification Codes

C43.

1. Introduction

Let p^0 and p^1 denote two positive vectors of prices pertaining to N commodities for periods 0 and 1 and let q^0 and q^1 denote the corresponding positive quantity vectors. Traditional bilateral index number theory assumes that the *bilateral price index formula*, $P(p^0,p^1,q^0,q^1)$, a function of the four vectors of prices and quantities, satisfies a sufficient number of "reasonable" tests or properties so that the functional form for P is determined.[2] The word "bilateral" [3] refers to the assumption that the function P depends only on the data pertaining to the two situations or periods being compared; i.e., P is regarded as a function of the two sets of price and quantity vectors, p^0,p^1,q^0,q^1, that are to

[1] The author is indebted to Bert Balk for helpful comments and to the Social Sciences and Humanities Research Council of Canada for financial support.
[2] For surveys of the axiomatic approach see Eichhorn and Voeller (1976), Eichhorn (1978), Diewert (1992), Balk (1995) and von Auer (2001).
[3] Multilateral index number theory refers to the situation where there are more than two situations whose prices and quantities need to be aggregated.

be aggregated into a single number that summarizes the overall change in the N price ratios, $p_1^1/p_1^0,\ldots, p_N^1/p_N^0$.

The "traditional" approach to bilateral index number theory assumes that along with the price index $P(p^0,p^1,q^0,q^1)$, there is a companion quantity index $Q(p^0,p^1,q^0,q^1)$ such that the product of these two indexes equals the value ratio between the two periods. Thus, it is assumed that P and Q satisfy the following *product test*:[4]

(1) $p^{1T}q^1/p^{0T}q^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1)$.

Equation (1) means that as soon as the functional form for the price index P is determined, then (1) can be used to determine the functional form for the quantity index Q. However, a further advantage of assuming that the product test holds is that we can assume that the quantity index Q satisfies a "reasonable" property and then use (1) to translate this test on the quantity index into a corresponding test on the price index P.

However, the axiomatic approach that will be taken in this paper is somewhat different. Instead of treating prices and quantities as the primary independent variables in the bilateral index number formula, we will follow the example of Walsh (1901; 104-105) and treat the *prices* and *expenditure shares* as the primary independent variables. Thus one of Walsh's approaches to index number theory was an attempt to determine the "best" weighted average of the price relatives, $r_n \equiv p_n^1/p_n^0$, for $n = 1,\ldots,N$.[5] This is equivalent to using an axiomatic approach to try and determine the "best" index of the form $P(r,v^0,v^1)$, where v^0 and v^1 are the vectors of expenditures on the N commodities during periods 0 and 1.[6] However, initially, rather than starting with indexes of the form $P(r,v^0,v^1)$, indexes of the form $P(p^0,p^1,v^0,v^1)$ will be considered, since this framework will be more comparable to the traditional bilateral axiomatic framework. As will be seen below, if the invariance to changes in the units of measurement test is imposed on an index of the form $P(p^0,p^1,v^0,v^1)$, then $P(p^0,p^1,v^0,v^1)$ can be written in the form $P(r,v^0,v^1)$.

[4] Notation: $q >> 0_N$ means that each component of the vector q is positive; $q \geq 0_N$ means each component of q is nonnegative and $q > 0_N$ means $q \geq 0_N$ and $q \neq 0_N$. Finally, $p^Tq \equiv \sum_{n=1}^{N} p_nq_n$ denotes the inner product of the vectors p and q.

[5] Fisher also took this point of view when describing his approach to index number theory: "An index number of the prices of a number of commodities is an average of their price relatives. This definition has, for concreteness, been expressed in terms of prices. But in like manner, an index number can be calculated for wages, for quantities of goods imported or exported, and, in fact, for any subject matter involving divergent changes of a group of magnitudes. Again, this definition has been expressed in terms of time. But an index number can be applied with equal propriety to comparisons between two places or, in fact, to comparisons between the magnitudes of a group of elements under any one set of circumstances and their magnitudes under another set of circumstances." Irving Fisher (1922; 3). However, in setting up his axiomatic approach, Fisher imposed axioms on the price and quantity indexes written as functions of the two price vectors, p^0 and p^1, and the two quantity vectors, q^0 and q^1; i.e., he did not write his price index in the form $P(r,v^0,v^1)$ and impose axioms on indexes of this type. Of course, in the end, his ideal price index turned out to be the geometric mean of the Laspeyres and Paasche price indexes and each of these indexes can be written as expenditure share weighted averages of the N price relatives, $r_n \equiv p_n^1/p_n^0$.

[6] Chapter 3 in Vartia (1976) considered a variant of this axiomatic approach.

Recall that the product test can used in order to define the quantity index, $Q(p^0,p^1,q^0,q^1) \equiv V^1/V^0P(p^0,p^1,q^0,q^1)$, that corresponded to the bilateral price index $P(p^0,p^1,q^0,q^1)$. A similar product test holds in the present framework; i.e., given that the functional form for the price index $P(p^0,p^1,v^0,v^1)$ has been determined, then the corresponding *implicit quantity index* can be defined in terms of P as follows:

$$(2)\ Q(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^N v_n^1 / [\sum_{n=1}^N v_n^0\, P(p^0,p^1,v^0,v^1)].$$

In the traditional approach to bilateral index number theory, the price and quantity indexes $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ are determined *jointly*; i.e., not only were axioms imposed on $P(p^0,p^1,q^0,q^1)$ but they were also imposed on $Q(p^0,p^1,q^0,q^1)$ and the product test (1) was used to translate these tests on Q into tests on P. In what follows, only tests on $P(p^0,p^1,v^0,v^1)$ will be used in order to determine the "best" price index of this form. Thus there is a parallel theory for quantity indexes of the form $Q(q^0,q^1,v^0,v^1)$ where it is attempted to find the "best" value weighted average of the quantity relatives, q_n^1/q_n^0.[7]

For the most part, the tests which will be imposed on the price index $P(p^0,p^1,v^0,v^1)$ in this paper are counterparts to the tests that are imposed on the price index $P(p^0,p^1,q^0,q^1)$ in the traditional approach to bilateral index number theory. However, a few new tests are also introduced; see section 7 below.

It turns out that our "new" axioms will characterize the Törnqvist (1936) Theil (1967) index P_T defined as follows.[8]

$$(3)\ \ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N (1/2)(s_n^0 + s_n^1) \ln (p_n^1/p_n^0)$$

where s_n^t is the expenditure share of commodity n in period t for t = 0,1 and n = 1,...,N.

In sections 2-7 below, we list various tests that will be used to characterize the Törnqvist Theil formula, while section 8 states our main result. Section 9 concludes and a technical Appendix proves our main result.

2. Preliminary Tests

It will be assumed that every component of each price and value vector is positive; i.e., $p^t >> 0_N$ and $v^t >> 0_N$ for t = 0,1. If it is desired to set $v^0 = v^1$, the common expenditure vector is denoted by v; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

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[7] It turns out that the price index that corresponds to this "best" quantity index, defined as $P^*(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^N v_n^1/[\sum_{n=1}^N v_n^0\, Q(q^0,q^1,v^0,v^1)]$, will not equal the "best" price index, $P(p^0,p^1,v^0,v^1)$. Thus the axiomatic approach to be developed in this paper generates separate "best" price and quantity indexes whose product does not equal the value ratio in general. This is a disadvantage of this second axiomatic approach to bilateral indexes compared to the traditional bilateral approach.

[8] This formula was explicitly defined in Törnqvist and Törnqvist (1937).

The first two tests are straightforward counterparts to the corresponding tests in the traditional approach to bilateral index number theory.[9]

T1: *Positivity*: $P(p^0,p^1,v^0,v^1) > 0$.

T2: *Continuity*: $P(p^0,p^1,v^0,v^1)$ is a continuous function of its arguments.

T3: *Identity or Constant Prices Test*: $P(p,p,v^0,v^1) = 1$.

The identity test suggests that if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the value vectors are. Note that the two value vectors are allowed to be different in the above test.

3. Homogeneity Tests

The following four tests restrict the behavior of the price index P as the scale of any one of the four vectors p^0,p^1,v^0,v^1 changes.

T4: *Proportionality in Current Prices*: $P(p^0,\lambda p^1,v^0,v^1) = \lambda P(p^0,p^1,v^0,v^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. This test is the counterpart to the traditional proportionality in current prices test:[10]

(4) $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

T5: *Inverse Proportionality in Base Period Prices*:

$$P(\lambda p^0,p^1,v^0,v^1) = \lambda^{-1}P(p^0,p^1,v^0,v^1) \text{ for } \lambda > 0.$$

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0. This test is the counterpart to the traditional inverse proportionality test in base period prices:[11]:

(5) $P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1}P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

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[9] See Diewert (1992; 214-223) or Balk (1995).

[10] This traditional test was proposed by Walsh (1901, 385), Eichhorn and Voeller (1976, 24) and Vogt (1980, 68).

[11] Eichhorn and Voeller (1976; 28) suggested this traditional test.

The following two homogeneity tests can also be regarded as invariance tests.

T6: *Invariance to Proportional Changes in Current Period Values*:

$$P(p^0,p^1,v^0,\lambda v^1) = P(p^0,p^1,v^0,v^1) \text{ for all } \lambda > 0.$$

That is, if current period values are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 1 value vector v^1.

T7: *Invariance to Proportional Changes in Base Period Values*:

$$P(p^0,p^1,\lambda v^0,v^1) = P(p^0,p^1,v^0,v^1) \text{ for all } \lambda > 0.$$

That is, if base period values are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 0 value vector v^0.

T6 and T7 together impose the property that the price index P does not depend on the *absolute* magnitudes of the value vectors v^0 and v^1. Using test T6 with $\lambda = 1/\sum_{i=1}^N v_i^1$ and using test T7 with $\lambda = 1/\sum_{i=1}^N v_i^0$, it can be seen that P has the following property:

$$(6)\ P(p^0,p^1,v^0,v^1) = P(p^0,p^1,s^0,s^1)$$

where s^0 and s^1 are the vectors of expenditure shares for periods 0 and 1; i.e., the ith component of s^t is $s_i^t \equiv v_i^t/\sum_{k=1}^N v_k^t$ for t = 0,1. Thus the tests T6 and T7 imply that the price index function P is a function of the two price vectors p^0 and p^1 and the two vectors of expenditure shares, s^0 and s^1.

Walsh suggested the spirit of tests T6 and T7 as the following quotation indicates:

"What we are seeking is to average the variations in the exchange value of one given total sum of money in relation to the several classes of goods, to which several variations [i.e., the price relatives] must be assigned weights proportional to the relative sizes of the classes. Hence the relative sizes of the classes at both the periods must be considered." Correa Moylan Walsh (1901; 104).

Walsh also realized that weighting the ith price relative r_i by the arithmetic mean of the value weights in the two periods under consideration, $(1/2)[v_i^0 + v_i^1]$ would give too much weight to the expenditures of the period that had the highest level of prices:

"At first sight it might be thought sufficient to add up the weights of every class at the two periods and to divide by two. This would give the (arithmetic) mean size of every class over the two periods together. But such an operation is manifestly wrong. In the first place, the sizes of the classes at each period are reckoned in the money of the period, and if it happens that the exchange value of money has fallen, or prices in general have risen, greater influence upon the result would be given to the weighting of the second period; or if prices in general have fallen, greater influence would be given to the weighting of the first period. Or in a comparison between two countries, greater influence would be given to the weighting of the country with the higher level of prices. But it is plain that *the one period, or the one country, is as important, in our comparison between them, as the other, and the weighting in the averaging of their weights should really be even*." Correa Moylan Walsh (1901; 104-105).

As a solution to the above weighting problem, Walsh (1901; 202) (1921a; 97) proposed the following *geometric Walsh price index*:

$$(7)\ P_{GW}(p^0,p^1,v^0,v^1) \equiv \prod_{n=1}^{N} [p_n^1/p_n^0]^{w(n)}$$

where the nth weight in the above formula was defined as

$$(8)\ w(n) \equiv (v_n^0 v_n^1)^{1/2} / \sum_{i=1}^{N} (v_i^0 v_i^1)^{1/2} = (s_n^0 s_n^1)^{1/2} / \sum_{i=1}^{N} (s_i^0 s_i^1)^{1/2} \ ; \qquad n = 1,\ldots,N.$$

The second equation in (8) shows that Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ can also be written as a function of the expenditure share vectors, s^0 and s^1; i.e., $P_{GW}(p^0,p^1,v^0,v^1)$ is homogeneous of degree 0 in the components of the value vectors v^0 and v^1 and so $P_{GW}(p^0,p^1,v^0,v^1) = P_{GW}(p^0,p^1,s^0,s^1)$. Thus Walsh came very close to deriving the Törnqvist Theil index defined earlier by (3).[12]

4. Invariance and Symmetry Tests

The next five tests are *invariance* or *symmetry* tests. Fisher (1922; 62-63, 458-460) and Walsh (1921b; 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, 62-63) spoke of fairness but it is clear that he had symmetry properties in mind.

The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed.

T8: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},v^{0*},v^{1*}) = P(p^0,p^1,v^0,v^1)$$

where p^{t*} denotes a permutation of the components of the vector p^t and v^{t*} denotes the same permutation of the components of v^t for t = 0,1. This test is the counterpart to a test due to Irving Fisher (1922), and it is one of his three famous reversal tests.

The next test asks that the index be invariant to changes in the units of measurement.

T9: *Invariance to Changes in the Units of Measurement* (commensurability test):

[12] One could derive Walsh's index using the same arguments as were used by Theil (1967; 136-137) in his stochastic approach, except that the geometric average of the expenditure shares $(s_n^0 s_n^1)^{1/2}$ could be taken as a preliminary probability weight for the nth logarithmic price relative, ln r_n. These preliminary weights are then normalized to add up to unity by dividing by their sum. It is evident that Walsh's geometric price index will closely approximate Theil's index using normal time series data. More formally, regarding both indexes as functions of p^0,p^1,v^0,v^1, it can be shown that $P_{GW}(p^0,p^1,v^0,v^1)$ approximates $P_T(p^0,p^1,v^0,v^1)$ to the second order around an equal price (i.e., $p^0 = p^1$) and quantity (i.e., $q^0 = q^1$) point. See Diewert (1978) for applications of similar approximation techniques.

$$P(\lambda_1 p_1^0,\ldots,\lambda_N p_N^0;\ \lambda_1 p_1^1,\ldots,\lambda_N p_N^1;\ v_1^0,\ldots,v_N^0;\ v_1^1,\ldots,v_N^1) =$$
$$P(p_1^0,\ldots,p_N^0;\ p_1^1,\ldots,p_N^1;\ v_1^0,\ldots,v_N^0;\ v_1^1,\ldots,v_N^1) \text{ for all } \lambda_1 > 0, \ldots, \lambda_N > 0.$$

That is, the price index does not change if the units of measurement for each commodity are changed. Note that the expenditure on commodity i during period t, v_i^t, does not change if the unit by which commodity i is measured changes. The concept of this test was due to Jevons (1884; 23) and the Dutch economist Pierson (1896; 131), who criticized several index number formula for not satisfying this fundamental test. Fisher (1911; 411) first called this test *the change of units test* and later, Fisher (1922; 420) called it the *commensurability test*.

The last test has a very important implication. Let $\lambda_1 = 1/p_1^0$, … , $\lambda_N = 1/p_N^0$ and substitute these values for the λ_i into the definition of the test. The following equation is obtained:

$$(9)\ P(p^0,p^1,v^0,v^1) = P(1_N,r,v^0,v^1) \equiv P^*(r,v^0,v^1)$$

where 1_N is a vector of ones of dimension N and r is a vector of the price relatives; i.e., the ith component of r is $r_i \equiv p_i^1/p_i^0$. Thus if the commensurability test T9 is satisfied, then the price index $P(p^0,p^1,v^0,v^1)$, which is a function of 4N variables, can be written as a function of 3N variables, $P^*(r, v^0,v^1)$, where r is the vector of price relatives and $P^*(r, v^0,v^1)$ is defined as $P(1_N,r,v^0,v^1)$.

The next test asks that the formula be invariant to the period chosen as the base period.

T10: *Time Reversal Test*: $P(p^0,p^1,v^0,v^1) = 1/P(p^1,p^0,v^1,v^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio, this test will be satisfied (as are all of the other tests listed in this section). The concept of the test was due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test, that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 368) (1921b; 541) and Fisher (1911; 534) (1922; 64).

The next test is a variant of the *circularity test*:[13]

T11: *Transitivity in Prices for Fixed Value Weights*:
$$P(p^0,p^1,v^r,v^s)P(p^1,p^2,v^r,v^s) = P(p^0,p^2,v^r,v^s).$$

In this test, the expenditure weighting vectors, v^r and v^s, are held constant while making all price comparisons. However, given that these weights are held constant, then the test

[13] The test name is due to Fisher (1922; 413) and the concept was originally due to Westergaard (1890; 218-219).

asks that the product of the index going from period 0 to 1, $P(p^0,p^1,v^r,v^s)$, times the index going from period 1 to 2, $P(p^1,p^2,v^r,v^s)$, should equal the direct index that compares the prices of period 2 with those of period 0, $P(p^0,p^2,v^r,v^s)$. Obviously, this test is a many commodity counterpart to a property that holds for a single price relative.

The final test in this section captures the idea that the value weights should enter the index number formula in a symmetric manner.

T12: *Quantity Weights Symmetry Test*: $P(p^0,p^1,v^0,v^1) = P(p^0,p^1,v^1,v^0)$.

That is, if the expenditure vectors for the two periods are interchanged, then the price index remains invariant. This property means that if values are used to weight the prices in the index number formula, then the period 0 values v^0 and the period 1 values v^1 must enter the formula in a symmetric or even handed manner.

5. Mean Value Tests

The next test is a *mean value test*.

T13: *Mean Value Test for Prices*:

$$\min_i (p_i^1/p_i^0 : i = 1,\ldots,N) \leq P(p^0,p^1,q^0,q^1) \leq \max_i (p_i^1/p_i^0 : i = 1,\ldots,N).$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is to be interpreted as an average of the n price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.[14]

6. Monotonicity Tests

The next two tests in this section are *monotonicity tests*; i.e., how should the price index $P(p^0,p^1,v^0,v^1)$ change as any component of the two price vectors p^0 and p^1 increases.

T14: *Monotonicity in Current Prices*: $P(p^0,p^1,v^0,v^1) < P(p^0,p^2,v^0,v^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase (holding the value vectors fixed), so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1 for fixed p^0, v^0 and v^1.

T15: *Monotonicity in Base Prices*: $P(p^0,p^1,v^0,v^1) > P(p^2,p^1,v^0,v^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0 for fixed p^1, v^0 and v^1.[15]

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[14] This test seems to have been first proposed by Eichhorn and Voeller (1976; 10).

[15] The traditional approach counterparts to T14 and T15 were proposed by Eichhorn and Voeller (1976; 23).

The above tests are not sufficient to determine the functional form of the price index; for example, it can be shown that both Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by (7) and the Törnqvist Theil index $P_T(p^0,p^1,v^0,v^1)$ defined by (3) satisfy all of the above axioms. Thus at least one more test will be required in order to determine the functional form for the price index $P(p^0,p^1,v^0,v^1)$.

7. Weighting Tests

The tests proposed thus far do not specify exactly how the expenditure share vectors s^0 and s^1 are to be used in order to weight say the first price relative, p_1^1/p_1^0. The next test says that only the expenditure shares s_1^0 and s_1^1 pertaining to the first commodity are to be used in order to weight the prices that correspond to commodity 1, p_1^1 and p_1^0.

T16. *Own Share Price Weighting*:

(10) $P(p_1^0,1,\ldots,1;p_1^1,1,\ldots,1;v^0;v^1) = f(p_1^0,p_1^1,v_1^0/\sum_{n=1}^N v_n^0,\ v_1^1/\sum_{n=1}^N v_n^1)$.

Note that $v_1^t/\sum_{k=1}^N v_k^t$ equals s_1^t, the expenditure share for commodity 1 in period t. The above test says that if all of the prices are set equal to 1 except the prices for commodity 1 in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity 1 and the two expenditure shares for commodity 1. The axiom says that a function of 2 + 2N variables is actually only a function of 4 variables.[16]

If test T16 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property:

(11) $P(1,\ldots,1,p_i^0,1,\ldots,1;1,\ldots,1,p_i^1,1,\ldots,1;v^0;v^1) = f(p_i^0,p_i^1,v_i^0/\sum_{n=1}^N v_n^0,\ v_i^1/\sum_{n=1}^N v_n^1)$; $i = 1,\ldots,N$.

Equation (11) says that if all of the prices are set equal to 1 except the prices for commodity i in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity i and the two expenditure shares for commodity i.

The final test that also involves the weighting of prices is the following one:

T17: *Irrelevance of Price Change with Tiny Value Weights*:

(12) $P(p_1^0,1,\ldots,1;\ p_1^1,1,\ldots,1;\ 0,v_2^0,\ldots,v_N^0;\ 0,v_2^1,\ldots,v_N^1) = 1$.

The test T17 says that if all of the prices are set equal to 1 except the prices for commodity 1 in the two periods, and the expenditures on commodity 1 are zero in the two periods but the expenditures on the other commodities are arbitrarily given, then the

[16] In the economics literature, axioms of this type are known as separability axioms.

index is equal to 1.[17] Thus, roughly speaking, if the value weights for commodity 1 are tiny, then it does not matter what the price of commodity 1 is during the two periods.

Of course, if test T17 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property: for i = 1,…,N:

(13)
$$P(1,\ldots,1,p_i^0,1,\ldots,1;1,\ldots,1,p_i^1,1,\ldots,1;v_1^0,\ldots,v_{i-1}^0,0,v_{i+1}^0,\ldots,v_N^0;v_1^1,\ldots,v_{i-1}^1,0,v_{i+1}^1,\ldots,v_N^1) = 1.$$

Equation (13) says that if all of the prices are set equal to 1 except the prices for commodity i in the two periods, and the expenditures on commodity i are 0 during the two periods but the other expenditures in the two periods are arbitrarily given, then the index is equal to 1.

This completes the listing of tests for the weighted average of price relatives approach to bilateral index number theory. It turns out that the above tests are sufficient to imply a specific functional form for the price index as will be seen in the next section.

8. The Törnqvist Theil Price Index and the Modified Walsh Test Approach to Bilateral Indexes

In the Appendix to this paper, it is shown that if the number of commodities N exceeds two and the bilateral price index function $P(p^0,p^1,v^0,v^1)$ satisfies the 17 axioms listed above , then P must be the Törnqvist Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined by (3).[18] Thus the 17 properties or tests listed above provide an axiomatic characterization of the Törnqvist Theil price index, just as the 20 tests listed in Diewert (1992; 214-222) provided an axiomatic characterization for the Fisher ideal price index.

Obviously, there is a parallel axiomatic theory for quantity indexes of the form $Q(q^0,q^1,v^0,v^1)$ that depend on the two quantity vectors for periods 0 and 1, q^0 and q^1, as well as on the corresponding two expenditure vectors, v^0 and v^1. Thus if $Q(q^0,q^1,v^0,v^1)$ satisfies the quantity counterparts to tests T1 to T17, then Q must be equal to the Törnqvist Theil quantity index $Q_T(q^0,q^1,v^0,v^1)$, whose logarithm is defined as follows:

$$(14)\ \ln Q_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N (1/2)(s_n^0 + s_n^1) \ln (q_n^1/q_n^0)$$

where as usual, the period t expenditure share on commodity i, s_i^t, is defined as $v_i^t/\sum_{k=1}^N v_k^t$ for i = 1,…,N and t = 0,1.

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[17] Strictly speaking, since all prices and values are required to be positive, the left hand side of (13) should be replaced by the limit as the commodity 1 values, v_1^0 and v_1^1, approach 0.

[18] The Törnqvist Theil price index satisfies all 17 tests but the proof in the Appendix did not use all of these tests to establish the result in the opposite direction: tests 5, 13, 15 and one of 10 or 12 were not required in order to show that an index satisfying the remaining tests must be the Törnqvist Theil price index. For alternative characterizations of the Törnqvist Theil price index, see Balk and Diewert (2001) and Hillinger (2002).

Unfortunately, the implicit Törnqvist Theil price index, $P_{IT}(q^0,q^1,v^0,v^1)$ that corresponds to the Törnqvist Theil quantity index Q_T defined by (14) using the product test is *not* equal to the direct Törnqvist Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined by (3). The product test equation that defines P_{IT} in the present context is given by the following equation:

(15) $P_{IT}(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^{N} v_n^1 / [\sum_{n=1}^{N} v_n^0 \, Q_T(q^0,q^1,v^0,v^1)]$.

The fact that the direct Törnqvist Theil price index P_T is not in general equal to the implicit Törnqvist Theil price index P_{IT} defined by (15) is a bit of a disadvantage compared to the traditional axiomatic approach to bilateral index number theory, which led to the Fisher ideal price and quantity indexes as being "best". Using the traditional Fisher approach meant that it was not necessary to decide whether one wanted a "best" price index or a "best" quantity index: the traditional theory determined both indexes simultaneously. However, in the Törnqvist Theil approach outlined in this paper, it is necessary to *choose* whether one wants a "best" price index or a "best" quantity index.[19]

Other tests are of course possible. A counterpart to the Paasche (1874) and Laspeyres (1871) bounding test in traditional index number theory,[20] is the following *geometric Paasche and Laspeyres bounding test*:

(16) $P_{GL}(p^0,p^1,v^0,v^1) \leq P(p^0,p^1,v^0,v^1) \leq P_{GP}(p^0,p^1,v^0,v^1)$ or
$P_{PL}(p^0,p^1,v^0,v^1) \leq P(p^0,p^1,v^0,v^1) \leq P_{LP}(p^0,p^1,v^0,v^1)$

where the logarithms of the geometric Laspeyres and geometric Paasche price indexes, P_{GL} and P_{GP}, are defined as follows:

(17) $\ln P_{GL}(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^{N} s_n^0 \ln (p_n^1/p_n^0)$;

(18) $\ln P_{GP}(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^{N} s_n^1 \ln (p_n^1/p_n^0)$.

As usual, the period t expenditure share on commodity i, s_i^t, is defined as $v_i^t/\sum_{k=1}^{N} v_k^t$ for $i = 1,\ldots,N$ and $t = 0,1$.

It can be show that Törnqvist Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined by (3) satisfies the geometric Laspeyres and Paasche bounding test but the geometric Walsh price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by (7) does not satisfy it.

The geometric Paasche and Laspeyres bounding test was not included as a primary test in section 5 because a priori, it was not known what form of averaging of the price relatives (e. g., geometric or arithmetic or harmonic) would turn out to be appropriate in this test framework. The test (16) is an appropriate one if it has been decided that geometric

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[19] Hillinger (2002) suggested taking the geometric mean of the direct and implicit Törnqvist Theil price indexes in order to resolve this conflict. Unfortunately, the resulting index is not "best" for either set of axioms that were suggested in this section.

[20] See Diewert (1992; 219) for this test in the traditional bilateral context.

averaging of the price relatives is the appropriate framework, since the geometric Paasche and Laspeyres indexes correspond to "extreme" forms of value weighting in the context of geometric averaging and it is natural to require that the "best" price index lie between these extreme indexes.

Walsh (1901; 408) pointed out a problem with his geometric price index P_{GW} defined by (7), which also applies to the Törnqvist Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined by (3): these geometric type indexes do not give the "right" answer when the quantity vectors are constant (or proportional) over the two periods. In this case, Walsh thought that the "right" answer must be the *Lowe* (1823) *index*, which is the ratio of the costs of purchasing the constant basket during the two periods. Put another way, the geometric indices P_{GW} and P_T do not satisfy the following fixed basket or constant quantities test for the traditional approach:[21]

(19) $P(p^0,p^1,q,q) = \sum_{i=1}^N p_i^1 q_i / \sum_{i=1}^N p_i^0 q_i$.

What then was the argument that led Walsh to define his geometric average type index P_{GW}? It turns out that he was led to this type of index by considering another test, which will now be explained.

Walsh (1901; 228-231) derived his test by considering the following very simple framework. Let there be only two commodities in the index and suppose that the expenditure share on each commodity is equal in each of the two periods under consideration. The price index under these conditions is equal to $P(p_1^0,p_2^0;p_1^1,p_2^1;v_1^0,v_2^0;v_1^1,v_2^1) = P^*(r_1,r_2;1/2,1/2;1/2,1/2) \equiv m(r_1,r_2)$ where $m(r_1,r_2)$ is a symmetric mean of the two price relatives, $r_1 \equiv p_1^1/p_1^0$ and $r_2 \equiv p_2^1/p_2^0$.[22] In this framework, Walsh then proposed the following *price relative reciprocal test*:

(20) $m(r_1,r_1^{-1}) = 1$.

Thus if the value weighting for the two commodities is equal over the two periods and the second price relative is the reciprocal of the first price relative r_1, then Walsh (1901; 230) argued that the overall price index under these circumstances ought to equal one, since the relative fall in one price is exactly counterbalanced by a rise in the other and both commodities have the same expenditures in each period. He found that the geometric mean satisfied this test perfectly but the arithmetic mean led to index values greater than one (provided that r_1 was not equal to one) and the harmonic mean led to index values

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[21] The origins of this test go back at least two hundred years to the Massachusetts legislature which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution; see Willard Fisher (1913). Other researchers who have suggested the test over the years include: Lowe (1823, Appendix, 95), Scrope (1833, 406), Jevons (1865), Sidgwick (1883, 67-68), Edgeworth (1925, 215) originally published in 1887, Marshall (1887, 363), Pierson (1895, 332), Walsh (1901, 540) (1921b; 544), and Bowley (1901, 227). Vogt and Barta (1997; 49) correctly observe that this test is a special case of Fisher's (1911; 411) proportionality test for quantity indexes which Fisher (1911; 405) translated into a test for the price index using the product test (1).

[22] Walsh considered only the cases where m was the arithmetic, geometric and harmonic means of r_1 and r_2.

that were less than one, a situation which was not at all satisfactory.[23] Thus he was led to some form of geometric averaging of the price relatives in one of his approaches to index number theory.

A generalization of Walsh's result is easy to obtain. Suppose that the mean function, $m(r_1,r_2)$, satisfies Walsh's reciprocal test, (20), and in addition, m is a homogeneous mean, so that it satisfies the following property for all $r_1 > 0$, $r_2 > 0$ and $\lambda > 0$:

(21) $m(\lambda r_1,\lambda r_2) = \lambda m(r_1,r_2)$.

Let $r_1 > 0$, $r_2 > 0$. Then

(22) $m(r_1,r_2) = [r_1/r_1]\, m(r_1,r_2)$
$= r_1\, m(r_1/r_1,r_2/r_1)$ using (21) with $\lambda \equiv 1/r_1$
$= r_1\, m(1,r_2/r_1)$
$= r_1\, f(r_2/r_1)$

where the function of one (positive) variable f(z) is defined as

(23) $f(z) \equiv m(1,z)$.

Using (20):

(24) $1 = m(r_1,r_1^{-1})$
$= [r_1/r_1]\, m(r_1,r_1^{-1})$
$= r_1\, m(1,r_1^{-2})$ using (21) with $\lambda \equiv 1/r_1$.

Using definition (23), (24) can be rearranged into the following equation:

(25) $f(r_1^{-2}) = r_1^{-1}$.

Letting $z \equiv r_1^{-2}$ so that $z^{1/2} = r_1^{-1}$, (24) becomes:

(26) $f(z) = z^{1/2}$.

Now substitute (26) into (22) and the functional form for the mean function $m(r_1,r_2)$ is determined:

(27) $m(r_1,r_2) = r_1 f(r_2/r_1) = r_1(r_2/r_1)^{1/2} = r_1^{1/2}r_2^{1/2}$.

Thus the geometric mean of the two price relatives is the only homogeneous mean that will satisfy Walsh's price relative reciprocal test.

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[23] "This tendency of the arithmetic and harmonic solutions to run into the ground or to fly into the air by their excessive demands is clear indication of their falsity." Correa Moylan Walsh (1901; 231).

There is one additional test that should be mentioned. Fisher (1911; 401) introduced this test in his first book that dealt with the test approach to index number theory. He called it the *test of determinateness as to prices* and described it as follows:

"A price index should not be rendered zero, infinity, or indeterminate by an individual price becoming zero. Thus, if any commodity should in 1910 be a glut on the market, becoming a 'free good', that fact ought not to render the index number for 1910 zero." Irving Fisher (1911; 401).

In the present context, this test could be interpreted as the following one: if any single price p_i^0 or p_i^1 tends to zero, then the price index $P(p^0,p,v^0,v^1)$ should not tend to zero or plus infinity. However, with this interpretation of the test, which regards the values v_i^t as remaining constant as the p_i^0 or p_i^1 tends to zero, none of the commonly used index number formulae would satisfy this test. Hence this test should be interpreted as a test that applies to price indexes $P(p^0,p^1,q^0,q^1)$ of the traditional bilateral, which is how Fisher intended the test to apply. Thus Fisher's price determinateness test should be interpreted as follows: if any single price p_i^0 or p_i^1 tends to zero, then the price index $P(p^0,p,q^0,q^1)$ should not tend to zero or plus infinity. With this interpretation of the test, it can be verified that Laspeyres, Paasche and Fisher indexes satisfy this test but the Törnqvist Theil price index will not satisfy this test. Thus when using the Törnqvist Theil price index, *care must be taken to bound the prices away from zero in order to avoid a meaningless index number value*.

Walsh was aware that geometric average type indexes like the Törnqvist Theil price index P_T or Walsh's geometric price index P_{GW} defined by (7) become somewhat unstable[24] as individual price relatives become very large or small:

"Hence in practice the geometric average is not likely to depart much from the truth. Still, we have seen that when the classes [i. e., expenditures] are very unequal and the price variations are very great, this average may deflect considerably." Correa Moylan Walsh (1901; 373).

"In the cases of moderate inequality in the sizes of the classes and of excessive variation in one of the prices, there seems to be a tendency on the part of the geometric method to deviate by itself, becoming untrustworthy, while the other two methods keep fairly close together." Correa Moylan Walsh (1901; 404).

9. Conclusion

Weighing all of the arguments and tests presented in this paper, it seems that there may be a slight preference for the use of the Fisher ideal price index as a suitable target index for a user that wishes to use the axiomatic approach, but of course, opinions can differ on which set of axioms is the most appropriate to use in practice.

Appendix: The Törnqvist Theil Price Index and the Walsh Bilateral Test Approach

Define $r_i \equiv p_i^1/p_i^0$ for $i = 1,\ldots,N$. Using T1, T9 and (9), $P(p^0,p^1,v^0,v^1) = P^*(r, v^0,v^1)$. Using T6, T7 and (5):

[24] That is, the index may approach zero or plus infinity.

(A1) $P(p^0,p^1,v^0,v^1) = P^*(r,s^0,s^1)$

where s^t is the period t expenditure share vector for $t = 0,1$.

Let $x \equiv (x_1,\ldots,x_N)$ and $y \equiv (y_1,\ldots,y_N)$ be strictly positive vectors. The transitivity test T11 and (A1) imply that the function P* has the following property:

(A2) $P^*(x,s^0,s^1)\, P^*(y,s^0,s^1) = P^*(x_1y_1,\ldots,x_Ny_N,s^0,s^1)$.

Using T1, $P^*(r,s^0,s^1) > 0$ and using T14, $P^*(r, s^0,s^1)$ is strictly increasing in the components of r. The identity test T3 implies that

(A3) $P^*(1_N,s^0,s^1) = 1$

where 1_N is a vector of ones of dimension N. Using a result due to Eichhorn (1978; 66), it can be seen that these properties of P* are sufficient to imply that there exist positive functions $\alpha_i(s^0,s^1)$ for $i = 1,\ldots,N$ such that P* has the following representation:

(A4) $\ln P^*(r,s^0,s^1) = \sum_{i=1}^N \alpha_i(s^0,s^1) \ln r_i$.

The continuity test T2 implies that the positive functions $\alpha_i(s^0,s^1)$ are continuous. For $\lambda > 0$, the linear homogeneity test T4 implies that

$$
\begin{aligned}
\text{(A5)} \ln P^*(\lambda r,s^0,s^1) &= \ln \lambda + \ln P^*(r,s^0,s^1) \\
&= \sum_{i=1}^N \alpha_i(s^0,s^1) \ln \lambda r_i && \text{using (A4)} \\
&= \sum_{i=1}^N \alpha_i(s^0,s^1) \ln \lambda + \sum_{i=1}^N \alpha_i(s^0,s^1) \ln r_i \\
&= \sum_{i=1}^N \alpha_i(s^0,s^1) \ln \lambda + \ln P^*(r,s^0,s^1) && \text{using (A4).}
\end{aligned}
$$

Equating the right hand sides of the first and last lines in (A5) shows that the functions $\alpha_i(s^0,s^1)$ must satisfy the following restriction:

(A6) $\sum_{i=1}^N \alpha_i(s^0,s^1) = 1$

for all strictly positive vectors s^0 and s^1.

Using the weighting test T16 and the commodity reversal test T8, equations (13) hold. Equations (13) combined with the commensurability test T9 implies that P* satisfies the following equations:

(A7) $P^*(1,\ldots,1,r_i,1,\ldots,1;s^0;s^1) = f(1,s_i^0,s_i^1)$; $\qquad i = 1,\ldots,N$

for all $r_i > 0$ where f is the function defined in test T16.

Substitute equations (A7) into equations (A4) in order to obtain the following system of equations:

(A8) $\ln P^*(1,\ldots,1,r_i,1,\ldots,1;s^0;s^1) = \ln f(1,s_i^0,s_i^1) = \alpha_i(s^0,s^1) \ln r_i$; $\qquad i = 1,\ldots,N$

But equation i in (A8) implies that the positive continuous function of 2N variables $\alpha_i(s^0,s^1)$ is constant with respect to all of its arguments except s_i^0 and s_i^1 and this property holds for each i. Thus each $\alpha_i(s^0,s^1)$ can be replaced by the positive continuous function of two variables $\alpha_i(s_i^0,s_i^1)$ for $i = 1,\ldots,N$.[25] Now replace the $\alpha_i(s^0,s^1)$ in equation (A4) by the $\alpha_i(s_i^0,s_i^1)$ for $i = 1,\ldots,N$ and the following representation for P* is obtained:

(A9) $\ln P^*(r,s^0,s^1) = \sum_{i=1}^N \alpha_i(s_i^0,s_i^1) \ln r_i$

Equations (A6) imply that the functions $\alpha_i(s_i^0,s_i^1)$ also satisfy the following restrictions:

(A10) $\sum_{n=1}^N s_n^0 = 1$; $\sum_{n=1}^N s_n^1 = 1$ implies $\sum_{i=1}^N \alpha_i(s_i^0,s_i^1) = 1$.

Assume that the weighting test T17 holds and substitute equations (13) into (A9) in order to obtain the following equations:

(A11) $\alpha_i(0,0) \ln [p_i^1/p_i^0] = 0$; $\qquad i = 1,\ldots,N.$

Since the p_i^1 and p_i^0 can be arbitrary positive numbers, it can be seen that (A11) implies

(A12) $\alpha_i(0,0) = 0$; $\qquad i = 1,\ldots,N.$

Assume that the number of commodities N is equal to or greater than 3. Using (A10) and (A12), Theorem 2 in Aczél (1987; 8) can be applied and the following functional form for each of the $\alpha_i(s_i^0,s_i^1)$ is obtained:

(A13) $\alpha_i(s_i^0,s_i^1) = \alpha s_i^0 + (1-\alpha)s_i^1$; $\qquad i = 1,\ldots,N$

where α is a positive number satisfying $0 < \alpha < 1$.

Finally, the time reversal test T10 *or* the quantity weights symmetry test T12 can be used to show that α must equal _. Substituting this value for α back into (A13) and then substituting those equations back into (A9), the functional form for P* and hence P is determined as

(A14) $\ln P(p^0,p^1,v^0,v^1) = \ln P^*(r,s^0,s^1) = \sum_{n=1}^N (1/2)[s_n^0 + s_n^1] \ln (p_n^1/p_n^0)$.

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[25] More explicitly, $\alpha_1(s_1^0,s_1^1) \equiv \alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ and so on. That is, in defining $\alpha_1(s_1^0,s_1^1)$, the function $\alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ is used where all components of the vectors s^0 and s^1 except the first are set equal to an arbitrary positive number like 1.

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